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INTERACTION OF DISPLACEMENT WAVES WITH CURVILINEAR LONGITUDINAL SHEAR CRACKS IN A PIEZOELECTRIC MEDIUM*

L.A. FIL'SHTINSKII and M.L. FIL'SHTINSKII

A dynamic antiplane problem of electroelasticity is considered for a piezoelectric medium with tunnel-type crack-cuts. The corresponding boundary value problem is reduced to a singular integro-differential equation for the displacement jumps at the cracks. An asymptotic expression is constructed for the coupled mechanical and electromagnetic fields near the singularities. Results of numerical solutions of the algorithm and some qualitative conclusions are given.

A dynamic antiplane problem was investigated in /1/ for a piezoelectric medium with a rectilinear tunnel cut, using the method of series.

1. Consider a transversely isotropic piezoelectric medium referred to the crystallographic axes x, y, z (the crystal is the hexagonal system 6mm and the piezoceramic is polarized), weakened by tunnel cuts L_j $(j = 1, 2, \ldots, k)$ along the z axis. Let a stress vector $X_n^\pm = 0$, $Y_n^\pm = 0$, $Z_n^\pm = \mathrm{Re}\,[Z^\pm e^{-i\omega t}]$, constant along the z axis and varying sinusoidally with time be given on the surfaces of the cavity-cuts, and let a monochromatic shear wave (Fig.1) be emitted from infinity

$$w_0 = \mathrm{Re}\,[W_0(x, y)\, e^{-i\omega t}] \tag{1.1}$$

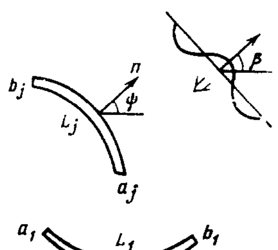

Fig.1

We assume that L_j $(j = 1, 2, \ldots, k)$ are simple open arcs with curvatures satisfying the condition H, and $Z^+ = -Z^- = Z$ are functions of class H on $L = \bigcup L_j/2/$.

Under these conditions, coupled mechanical and electromagnetic fields appear in the medium, corresponding to the state to antiplane deformation. The complete system of equations has the form /3/:

$$\begin{aligned} &\tau_{xz} = c_{44}\partial_1 w - e_{15}E_x, \quad D_x = e_{15}\partial_1 w + \varepsilon_{11}E_x \\ &\tau_{yz} = c_{44}\partial_2 w - e_{15}E_y, \quad D_y = e_{15}\partial_2 w + \varepsilon_{11}E_y \\ &\partial_1 = \partial/\partial x, \ \partial_2 = \partial/\partial y \end{aligned} \tag{1.2}$$

$$\partial_1\tau_{xz} + \partial_2\tau_{yz} = \rho\frac{\partial^2 w}{\partial t^2} \tag{1.3}$$

$$\begin{aligned} &\partial_1 E_y - \partial_2 E_x + \frac{\mu_0\mu}{c}\frac{\partial H_z}{\partial t} = 0, \quad \partial_1 D_x + \partial_2 D_y = 0 \\ &\partial_2 H_z = \frac{1}{c}\frac{\partial D_x}{\partial t}, \quad \partial_1 H_z = -\frac{1}{c}\frac{\partial D_y}{\partial t} \end{aligned} \tag{1.4}$$

Here (1.2) are the equations of state, (1.3) the equations of motion and (1.4) Maxwell equations, τ_{xz}, τ_{yz} and w are shear stresses and displacements along the z axis, E_x, E_y, H_z and D_x, D_y are the corresponding electric and magnetic field strength components and electric induction vector components, c_{44} is the shear modulus, e_{15} is the piezoelectric constant, and ε_{11} and μ are the permittivity and permeability of the medium. We assume that there are no external charges and the conductivity of the medium is equal to zero.

The electric and magnetic boundary conditions at the cut edges are taken in the form

$$E_s^- = E_s^-, \quad D_n^+ = D_n^-, \quad H_s^+ = H_s^-, \quad B_n^+ = B_n^-, \quad B = \mu\mu_0 H \tag{1.5}$$

Here E_s and H_s are the tangential components of the electric and magnetic field vectors, and D_n and B_n are the normal components of the electric and magnetic induction vectors.

Henceforth, all calculations will be carried out in the electromagnetic system of units.

Introducing the function Φ according to the formulas /1/

$$E_x = -\frac{e_{15}}{\varepsilon_{11}}\partial_1 w + \partial_2\Phi, \quad E_y = -\frac{e_{15}}{\varepsilon_{11}}\partial_2 w - \partial_1\Phi, \quad H_z = \varepsilon_{11}\frac{\partial\Phi}{\partial t} \tag{1.6}$$

we arrive at the expressions

$$\nabla^2 w - \frac{1}{a^2}\frac{\partial^2 w}{\partial t^2} = 0, \quad \nabla^2\Phi - \frac{1}{c_\alpha^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \tag{1.7}$$

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$$a=\sqrt{\frac{c_{44}}{\rho}(1+\varkappa_0^2)},\quad \varkappa_0^2=\frac{e_{15}^2}{\varepsilon_{11}c_{44}},\quad c_\alpha^2=\frac{1}{\varepsilon_{11}\mu_0\mu}$$

For example, for the ceramic PZT-4, $c_\alpha\approx 1{,}11\cdot 10^7$ m/sec. Therefore, when the cuts are not very large, we can assume that $\nabla^2\Phi=0$.

By virtue of (1.1), (1.2), (1.6) we have

$$\tau_{xz}=c_{44}(1+\varkappa_0^2)\,\partial_1 w-e_{15}\partial_2\Phi$$
$$\tau_{yz}=c_{44}(1+\varkappa_0^2)\,\partial_2 w+e_{15}\partial_1\Phi,\ w=w_0+w_1$$
$$D_x=\varepsilon_{11}\partial_2\Phi,\ D_y=-\varepsilon_{11}\partial_1\Phi,\ D_z=0$$

Here w_1 characterizes the perturbations in the displacement field due to the cuts.

Putting (β is the angle between the normal to the wave front and the Ox axis)

$$w=\mathrm{Re}\,[W(x,y)\,e^{-i\omega t}],\ \Phi=\mathrm{Re}\,[F(x,y)\,e^{-i\omega t}]$$
$$W=W_0+W_1,\ W_0=\tau e^{-i(\alpha_1x+\alpha_2y)}$$
$$\alpha_1=\gamma_2\cos\beta,\ \alpha_2=\gamma_2\sin\beta,\ \gamma_2=\omega/a$$

we can write the boundary conditions at the edges L_j in the form

$$c_{44}(1+\varkappa_0^2)\left\{e^{i\psi}\left(\frac{\partial W}{\partial\zeta}\right)^{\pm}+e^{-i\psi}\left(\frac{\partial W}{\partial\bar\zeta}\right)^{\pm}\right\}- \tag{1.8}$$
$$ie_{15}\left\{e^{i\psi}\left(\frac{\partial F}{\partial\zeta}\right)^{\pm}-e^{-i\psi}\left(\frac{\partial F}{\partial\bar\zeta}\right)^{\pm}\right\}=\pm Z^{\pm}$$
$$i\frac{e_{15}}{\varepsilon_{11}}\left\{e^{i\psi}\left[\frac{\partial W}{\partial\zeta}\right]-e^{-i\psi}\left[\frac{\partial W}{\partial\bar\zeta}\right]\right\}+e^{-i\psi}\left[\frac{\partial F}{\partial\bar\zeta}\right]+e^{i\psi}\left[\frac{\partial F}{\partial\zeta}\right]=0$$
$$e^{i\psi}\left[\frac{\partial F}{\partial\zeta}\right]-e^{-i\psi}\left[\frac{\partial F}{\partial\bar\zeta}\right]=0$$
$$\zeta=\xi+i\eta,\ \bar\zeta=\xi-i\eta,\ \zeta\in L_j,\ [f]=f^+-f^-\ (j=1,2,\ldots,k)$$

The upper sign refers to the left edge of L_j (on moving from its beginning a_j to the end b_j), ψ is the angle between the positive direction of the normal to the left edge and the Ox axis. The continuity conditions to the magnetic quantities across L_j are satisfied automatically.

2. To derive the integral equation for the boundary value problem stated, we must construct the integral representations for the functions W and F. This is easily done using standard methods of potential theory. In the present case, however, we would have to apply the procedure for the regularization of divergent integrals /4/. To avoid this, we shall construct integral representations not of the functions themselves, but of their first-order derivatives.

In addition to the fundamental state of the system we will introduce an auxilliary state characterized by the presence, at some inner point of the region (x_0, y_0), of a concentrated functional $Q\delta(x-x_0, y-y_0)$ where Q is the strength of the forces per unit length concentrated on the line $x=x_0\ y=y_0,\ -\infty<z<\infty$

Let us calculate the sums of products of the Helmholtz equation for the i-th state and the corresponding derivative of the amplitude of the displacements of the j-th state ($i\neq j$; $i, j=1,2$). This yields a divergence-type equation. Carrying out the integration over the region occupied by the body and applying Green's formula, we obtain

$$\frac{\partial W_1}{\partial z}=\frac{c_{44}}{Q}\int_L\left\{\frac{\gamma_2^2}{2}e^{-i\psi}Eq+ipe^{i\psi}\frac{\partial E}{\partial\zeta}\right\}ds \tag{2.1}$$
$$\frac{\partial W_1}{\partial \bar z}=\frac{c_{44}}{Q}\int_L\left\{\frac{\gamma_2^2}{2}e^{i\psi}Eq-ip_1e^{-i\psi}\frac{\partial E}{\partial\bar\zeta}\right\}ds$$
$$p=p(\zeta)=-2i\left[\frac{\partial W_1}{\partial\zeta}\right],\quad p_1=p_1(\zeta)=2i\left[\frac{\partial W_1}{\partial\bar\zeta}\right]$$
$$q=q(\zeta)=[W_1],\quad E=E(x-\xi,y-\eta)=-\frac{iQ}{4c_{44}}H_0^{(1)}(\gamma_2r)$$
$$r=|z-\zeta|,\ z=x+iy,\ \zeta=\xi+i\eta\in L$$

Here $[\partial W_1/\partial\zeta]$, $[\partial W_1/\partial\bar\zeta]$ and $[W_1]$ are the jumps in the values of the corresponding quantities on L, ds is the element of the arc of the contour L, $H_n^{(1)}(x)$ is the n-th order Hankel function of the first kind, and the derivatives of the functions sought are determined at the inner point of the region (x, y).

The representations (2.1) satisfy the radiation condition /5/ and agree with the integral representation for W_1 of the form

$$W_1(x,y)=c_{44}(1+\varkappa_0^2)\int_L\left\{q\left(e^{i\psi}\frac{\partial E}{\partial\zeta}+e^{-i\psi}\frac{\partial E}{\partial\bar\zeta}\right)-\frac{i}{2}(pe^{i\psi}-p_1e^{-i\psi})E\right\}ds$$

We will write the function F as follows:

$$F(x,y)=\frac{1}{2\pi i}\int_L f(\zeta)\ln(\zeta-z)\,d\zeta-\frac{1}{2\pi i}\int_L \overline{f_1(\zeta)}\ln(\bar\zeta-\bar z)\,d\bar\zeta$$

The meaning of the functions f and f_1 will be clarified below.

Substituting the limit values of the functions (2.1) and the derivatives $\partial F/\partial z$ and $\partial F/\partial\bar z$ as $z\to\zeta_0\in L$ into the boundary conditions (1.8), we obtain

$$p(\zeta)=-e^{-i\psi}\left[\frac{\partial W_1}{\partial s}\right],\quad p_1(\zeta)=-e^{i\psi}\left[\frac{\partial W_1}{\partial s}\right]$$
$$f(\zeta)=-\frac{e_{15}}{2\varepsilon_{11}}p(\zeta),\quad \overline{f_1(\zeta)}=-\frac{e_{15}}{2\varepsilon_{11}}p_1(\zeta) \tag{2.2}$$

The mechanical boundary condition in (1.8), taking (2.2) into account, leads to the following singular integro-differential equation:

$$\int_L\left[\frac{\partial W_1}{\partial s}\right]g(\zeta,\zeta_0)\,ds+\int_L[W_1]\,G(\zeta,\zeta_0)\,ds=N(\zeta_0) \tag{2.3}$$
$$g(\zeta,\zeta_0)=\frac{1}{\pi i}\operatorname{Re}\left(\frac{i\exp(i\psi_0)}{\zeta-\zeta_0}\right)+\frac{\gamma_2}{2}(1+\varkappa_0^2)H_1(\gamma_2 r_0)\sin(\alpha_0-\psi_0)$$
$$G(\zeta,\zeta_0)=\frac{\gamma_2^2}{2}(1+\varkappa_0^2)H_0^{(1)}(\gamma_2 r_0)\cos(\psi-\psi_0)$$
$$N(\zeta_0)=\frac{2i}{c_{44}}Z(\zeta_0)-2\tau(1-\varkappa_0^2)\gamma_2\exp[-i(\alpha_1\xi_0+\alpha_2\eta_0)]\cos(\psi_0-\beta)$$
$$H_1(\gamma_2 r)=\frac{2i}{\pi\gamma_2 r}+H_1^{(1)}(\gamma_2 r),\quad r_0=|\zeta-\zeta_0|$$
$$\alpha_0=\arg(\zeta-\zeta_0),\quad \psi_0=\psi(\zeta_0),\quad \zeta_0=\xi_0+i\eta_0\in L_j\quad (j=1,2,\ldots,k)$$

To obtain a unique solution of Eq.(2.3) in the class h_0 /2/, we must add the conditions

$$\int_{L_j}\left[\frac{\partial W_1}{\partial s}\right]ds=0\qquad (j=1,2,\ldots,k) \tag{2.4}$$

3. Suppose that there is a single cut L in the medium whose parametric equation is $\xi=\xi(\delta),\ \eta=\eta(\delta)\ (-1\leqslant\delta\leqslant 1)$. In accordance with this we shall represent the solution of equations (2.3), (2.4) as follows:

$$\left[\frac{\partial W_1}{\partial s}\right]=\frac{\Omega_0(\delta)}{s'(\delta)\sqrt{1-\delta^2}},\quad s'(\delta)=\frac{ds}{d\delta},\quad \Omega_0(\delta)\in H[-1,1] \tag{3.1}$$

Asymptotic analysis of the representations (2.1) taking relations (1.2), (3.1) into account, yields the following expression for the stresses on the continuation beyond the tip of the cut:

$$\tau_{xz}+i\tau_{yz}=-c_{44}e^{i\psi(\mp1)}\frac{\operatorname{Re}\{e^{-i\omega t}\Omega_0(\mp1)\}}{2\sqrt{2rs'(\mp1)}} \tag{3.2}$$

Here r denotes the distance from the tip, and upper sign refers to the beginning, and the lower to the end of the cut.

The dynamic mechanical stress intensity coefficient /6/ is given by the formula

$$k_3=\sqrt{2\pi r}\,\tau_n=-\frac{c_{44}}{2}\sqrt{\frac{\pi}{s'(\mp1)}}\operatorname{Re}\{e^{-i\omega t}\Omega_0(\mp1)\} \tag{3.3}$$

The asymptotic form of the normal component of the electric induction vector on the continuation beyond the crack tip is

$$D_n=D_x\cos\psi(\mp1)+D_y\sin\psi(\mp1)=-e_{15}\frac{\operatorname{Re}\{e^{-i\omega t}\Omega_0(\mp1)\}}{2\sqrt{2rs'(\mp1)}} \tag{3.4}$$

The electric and magnetic field strength vectors are bounded. This is explained by the fact that in the static problem of electroelasticity dealing with the longitudinal shear of the transversely isotropic medium, the mechanical and electric fields are not connected; therefore there is no electric field when external mechanical forces act on the medium. Thus we find that under a mechanical load the singular part of the electric strength vector in the dynamic problem is equal to zero.

4. Eq.(2.3) together with the additional condition (2.4) was solved numerically using

the Multopp-type scheme /7/ for the case when the piezoelectric ceramic PZT-5 had a single parabolic crack $\xi = p_1\delta$, $\eta = p_2\delta^2$, $\delta \in [-1, 1]$. The approximate values of the function $\Omega_0(\delta)$ at the Chebyshev interpolation nodes were computed for a number of nodes equal to $n = 15, 21$ and 31. A further increase in the number of decompositions did not, in practice, improve the accuracy.

Let $\tau = 0$ (there is no radiation from infinity), and $Z = \mathrm{const}$. Fig.2 shows the variation in the relative quantity $\alpha^+ = c_{44} \cdot |\Omega_0(1)|/(2Z\sqrt{ls'(1)})$ relative to the normalized wave number $\gamma_2 l$ ($2l$ is the crack length) for $p_1 = 1$, using the solid lines 1, 2, 3 to represent the values of $p_2 = 0$; 0.5 and 1 respectively. Clearly, $\alpha^- = \alpha^+$.

Knowing the values of $\alpha^\mp$ and $\delta^\mp = \arg[-\Omega_0(\mp 1)]$, we can find the intensity coefficient k_3 using the formula

$$k_3 = Z\sqrt{\pi l}\alpha^\mp \cos(\omega t - \delta^\mp)$$

The variation in the relative magnitude of $\alpha^+ = c_{44}|\Omega_0(1)|/(2T_{yz}{}^\circ\sqrt{ls'(1)})$ for the case when the cut edges are free of forces ($Z = 0$), and a monochromatic shear wave ($\tau \neq 0$) is radiated from infinity along the y axis, is shown in Fig.2, the dashed lines *1, 2* and *3* corresponding to $p_2 = 0$; 0.5 and 1, respectively. Here $T_{yz}{}^\circ$ denotes the stress τ_{yz} amplitude modulus in the incident wave. The dynamic intensity coefficient k_3 is given in this case by the formula

$$k_3 = \sqrt{\pi l}T_{yz}{}^\circ\alpha^\mp \cos(\omega t - \delta^\mp)$$

From relations (3.2) and (3.3) it follows that the quantitiy $\sqrt{2\pi r}D_n$ is proportional to k_3 near the crack tip. The graphs of Fig.2 can be used to determine D_n.

Calculations show that when the crack curvature increases, $\max\alpha^+$ is displaced along $\gamma_2 l$ to the right and its magnitude increases in the first numerical example, and decreases in the second. In the latter case the local maxima of the functions $\alpha^+(\gamma_2 l) = \alpha^-(\gamma_2 l)$ alternate with local minima, and decrease as $\gamma_2 l$ increases.

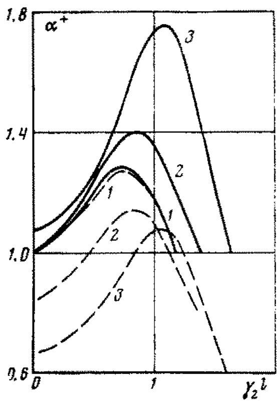

Fig.2

To estimate the influence of the coupled electromagnetic field on the mechanical stress intensity coefficient, we shall give the following data for the parabolic crack ($\xi = \delta$, $\eta = p_2\delta^2$, $-1 \leqslant \delta \leqslant 1$, $Z = \mathrm{const}$, $\tau = 0$):

The results on the left-hand side of the table correspond to an isotropic medium, and those on the right-hand side correspond to the piezoelectric ceramic $PZT = 5$.

p_2	0	0.5	1	0	0.5	1
$\gamma_2 l$	0.9	1.1	1.35	0.75	0.9	1.1
$\max\alpha^+$	1.20	1.30	1.65	1.27	1.40	1.76

The converse effect can be assumed to exist, since the mechanical external loads induce a singular electric displacement field.

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